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Lecture 14: Gap Amplification and the Detectability Lemma

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1 Introduction

First to get some intuition lets look at how gap amplification looks in the simple classical setting. Consider some Hamiltonian $H = \sum_i h_i$ where h_i are 2-local acting on n-particles each with dimension d. Lets consider different Hamiltonians we can map this to in the hopes of amplifying the gap.

1. **Parallel repetition:** We map to a new Hamiltonian $H' = \sum_i h_i \cap h_j$ where the terms are again 2-local but acting on n^2 particles each with dimension d^2.
2. **In-place parallel repetition:** We map to a new Hamiltonian $H' = \sum_{i,j} h_i \cup h_j$. In this setting if we compute $\langle Z|H|Z\rangle = \text{Prob}_i[h_i \text{ violated on Z}]$ and compare that to $\langle Z|H'|Z\rangle = \text{Prob}_{i,j}[h_i \text{ or } h_j \text{ violated on Z}]$ we see that $\langle Z|H'|Z\rangle \sim 2\langle Z|H|Z\rangle$.
3. **Dinur:** Pick $i_1, ..., i_t$ by walk on an expander.
4. **In-place Dinur:** Here we map to $H' = \sum_{i_1,...,i_t} h_{i_1} \cup ... \cup h_{i_t}$ where the i_j are chosen by a walk on an expander. With such a procedure one can get $\lambda_{min}(H') \geq t \cdot c \cdot \lambda_{min}(H)$ where c is a constant.

Going from the classical case to the quantum case there are a few details we should think about.

1. **Question 1:** What should we do about non-commuting constraints?
2. **Question 2:** What should we do about fractional violations?

The solution to the first question is that we take our local Hamiltonian and split it up into multiple layers where in each layer the operators all commute. Then we would consider counting the number of violations in each layer. For example, consider a system defined on a 1D chain with Hamiltonian $H = \sum_i h_i$, where h_i acts on site i and $i + 1$. We would break up the Hamiltonian into $H = H_{odd} + H_{even}$ where $H_{odd} = \sum_j h_{2j+1}$ and $H_{even} = \sum_j h_{2j}$. Clearly all the terms within H_{even} and H_{odd} commute since none of the terms overlap. By Vizing's theorem the same can be done on any graph with bounded degree.

To understand the second question imagine we had a state

$$|\psi\rangle = \sqrt{1 - \frac{\epsilon_0}{m}}|E_{odd} = 0\rangle + \sqrt{\frac{\epsilon_0}{m}}|E_{odd} = m\rangle \tag{1}$$

In this case even when measuring all of H_0 the probability we see no violations is still $1 - \frac{\epsilon_0}{m}$. In practice this may make gap amplification challenging to get working in practice.

2 Detectability Lemma

2.1 Statement

We use the version of the detectability lemma from [AAV16]. The statement is as follows.

Theorem 2.1. *Consider a Hamiltonian $H = \sum_{i=1}^{m} Q_i$ where each Q_i commutes with all but g of the Q_j. Given a state $|\psi\rangle$, let $|\phi\rangle = \prod_{i=1}^{m}(I - Q_i)|\psi\rangle$ and define $\epsilon_\phi = \frac{1}{||\phi||^2}\langle\phi|H|\phi\rangle \geq \lambda_{min}(H)$. Then we have that*

$$|||\phi\rangle||^2 = ||\prod_{i=1}^{m}(I - Q_i)|\psi\rangle||^2 \leq \frac{1}{\epsilon_0/g^2 + 1} \tag{2}$$

Proof. We prove a simpler version for the case of a frustration free Hamiltonian. To start we consider

$$\langle\phi|H|\phi\rangle \;=\; \sum_{i=1}^{m}\langle\phi|Q_i|\phi\rangle \;=\; \sum_{i=1}^{m}\big\|\, Q_i|\phi\rangle\,\big\|^2. \tag{3}$$

We focus on one i and expand using the definition of ϕ.

$$\big\|Q_i|\phi\rangle\big\| = \big\|\, Q_i(\mathbb{I} - Q_m)\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\|. \tag{4}$$

We let j be the first non-commuting term ($[Q_i, Q_j] \neq 0$. Then we get

$$\big\|Q_i|\phi\rangle\big\| = \big\|\, Q_i(\mathbb{I} - Q_m)\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\| \leq \big\|\, Q_i(\mathbb{I} - Q_j)\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\|. \tag{5}$$

Then using the triangle inequality we get

$$\big\|Q_i|\phi\rangle\big\| \leq \big\|\, Q_i(\mathbb{I} - Q_j)(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\,\big\| \tag{6}$$

$$\leq \big\|\, Q_i(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\,\big\| + \big\|\, Q_iQ_j(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\,\big\|. \tag{7}$$

Now we repeat this process iteratively until we get

$$\big\|Q_i|\phi\rangle\big\| \;\leq\; \sum_{j\in N_i}\big\|\, Q_j(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\|. \tag{8}$$

where N_i is the set of indices that Q_i doesn't commute with (and hence $|N_i| \leq g$). Now we go back to looking at $|Q_i|\phi\rangle\big\|^2$ and using Cauchy-Schwartz get the bound

$$\big\|Q_i|\phi\rangle\big\|^2 \;\leq\; g\sum_{j\in N_i}\big\|\, Q_j(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\|^2. \tag{9}$$

Now we sum over i and use that each j belongs to at most g different N_i.

$$\langle\phi|H|\phi\rangle = \sum_{i=1}^{m}\big\|Q_i|\phi\rangle\big\|^2 \tag{10}$$

$$\leq g^2\sum_{j=2}^{m}\big\|\, Q_j(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)\,|\psi\rangle\,\big\|^2. \tag{11}$$

Now we expand in the following way

$$\begin{aligned}\langle\phi|H|\phi\rangle &\leq g^2\sum_{j=2}^{m}\big\|\, Q_j(\mathbb{I}-Q_{j-1})\cdots(\mathbb{I}-Q_1)\,|\psi\rangle\,\big\|^2 \\ &= g^2\sum_{j=2}^{m} -\big\|\,(\mathbb{I}-Q_j)\cdots(\mathbb{I}-Q_1)\,|\psi\rangle\,\big\|^2 + \big\|\,(\mathbb{I}-Q_{j-1})\cdots(\mathbb{I}-Q_1)\,|\psi\rangle\,\big\|^2 \end{aligned} \tag{12}$$

Next we identify that the sum cleanly telecopes so we just end up with

$$\langle\phi|H|\phi\rangle = g^2\Big(\,\|(\mathbb{I}-Q_1)|\psi\rangle\|^2 - \|(\mathbb{I}-Q_m)\cdots(\mathbb{I}-Q_1)|\psi\rangle\|^2\Big) \tag{13}$$

$$= g^2\Big(\,\|(\mathbb{I}-Q_1)|\psi\rangle\|^2 - \|\phi\|^2\Big) \;\leq\; g^2\big(1-\|\phi\|^2\big), \tag{14}$$

where in the last inequality we used $\|(\mathbb{I}-Q_1)|\psi\rangle\| \leq \||\psi\rangle\| = 1$. Thus we get the desired expression. □

Some useful corollaries to the detectability lemma are included below.

Suppose $\lambda_{min}(H) = 0$, $\lambda_2(H) = \gamma$, and that $|\psi\rangle$ is orthogonal to the ground space. Then we have that

$$|||\phi\rangle||^2 = ||\prod_{i=1}^{m}(I-Q_i)|\psi\rangle||^2 \leq \frac{1}{\gamma/g^2+1} \tag{15}$$

2.2 A local Hamiltonian showing the tightness of DL

Note added by Anurag.

Consider the Hamiltonian on the star graph with n vertices connected to a center vertex: $H = \sum_{i=1}^{n}|\psi\rangle\langle\psi|_{0,i} + H_c$, where $|\psi\rangle = \frac{|0,1\rangle - b|1,0\rangle}{\sqrt{1+b^2}}$ and $H_c = \sum_{i=0}^{n}\sum_{j=i+1}^{n}|1,1\rangle\langle 1,1|_{i,j}$. The term H_c forces that the ground space of H is in the span of $|00\ldots0\rangle, |10\ldots0\rangle, \ldots |00\ldots1\rangle$.

Let $|\bar{i}\rangle$ be the 'single particle' state $|0\ldots01_i0\ldots0\rangle$. There are two ground states - $|00\ldots0\rangle$ (in the 'zero particle' sector) and

$$|gs_1\rangle = \frac{bn\,|\bar{0}\rangle + \sum_i|\bar{i}\rangle}{\sqrt{n+b^2n^2}}$$

(in the single particle sector). The spectral gap is $\frac{1}{1+b^2}$. Thus, the hamiltonian is gapped for all b, unlike the 1D model where gap vanishes when $b=1$.

We can consider the detectability lemma operator. Due to the contribution from H_c, the non-zero eigenvalues of the DL operator is only in the single or zero particle sector (more than one particles get annihilated by terms in DL from H_c). In the zero particle sector, there is only one state which is the ground state. Thus, we can focus on the one particle sector. Here, we find that DL is

$$DL_1 = \prod_{i=1}^{n}\left(1 - \frac{(|\bar{0}\rangle - b\,|\bar{i}\rangle)(\langle\bar{0}| - b\,\langle\bar{i}|)}{1+b^2}\right) = \prod_{i=1}^{n}\left(\frac{(b\,|\bar{0}\rangle + |\bar{i}\rangle)(b\,\langle\bar{0}| + \langle\bar{i}|)}{1+b^2}\right).$$

We can explicitly compute the action of this operator on the state $|\phi\rangle = \frac{\sum_{i=0}^{n}|\bar{i}\rangle}{\sqrt{n}}$. We find that $\|DL_1\,|\phi\rangle\,\|^2 = 1 - \frac{f(b)}{n}$, where $f(b)$ is some simple function of b (this was calculated by ChatGPT).

This means that,

$$\|DL_1(|\phi\rangle - \langle\phi|gs_1\rangle\,|gs_1\rangle)\| \geq 1 - \frac{f(b)}{n} - |\langle\phi|gs_1\rangle| = 1 - \frac{f(b)}{n} - \frac{(1+b)}{\sqrt{1+b^2n}} = 1 - \frac{\Theta(1)}{\sqrt{n}}.$$

The vector $|\phi\rangle - \langle\phi|gs_1\rangle\,|gs_1\rangle$ is orthogonal to the ground state and has norm $1 - |\langle\phi|gs_1\rangle|^2 = 1 - \frac{\Theta(1)}{n}$. This shows that the degree dependence in DL is necessary (which is n in this case). Note that we can take many copies of the system to make the degree n independent of the total number of qubits and the total norm of the Hamiltonian. Note above that we need $b > 0$ as $f(b) = \Omega(\frac{1}{b})$, but otherwise b can be any constant, such as 1.

3 Gap Amplification

Now we discuss the gap amplification procedure of [AALV08]. First we introduce the procedure and next we show how it be be applied to give a amplification in the limit of a small number of violated constraints.

Again the starting point is a Hamiltonian $H = \sum_i Q_i$. We consider the following transformation

$$H' = \frac{1}{\#\text{ t-walks}} \sum_{\vec{e}:\text{expander walk}} Q_{\vec{e}}, \quad 1 - Q_{\vec{e}} = \text{proj onto intersection of } 1 - Q_i \ \forall i \in \vec{e} \tag{16}$$

We apply this transformation to each layer of our Hamiltonian. More formally we take $H_i' = (H_i)'$ where all but the ith layer constraints are replaced by 0. In other words the ith layer of the new Hamiltonian is the transformed version of the ith layer of the original Hamiltonian. So our original Hamiltonian which takes the form $H = \sum_{i=1}^{L} H^{(i)}$ becomes

$$H' = \sum_{i=1}^{L} (H^{(i)})' \tag{17}$$

Note that $QUNSAT(H') \geq QUNSAT(H_i')$ for all i. Thus if we can lower bound $QUNSAT(H_i')$ for some i we will have a lower bound on $QUNSAT(H')$.

In order to get a lower bound on $QUNSAT(H_i')$ we make use of the fact that $(H^{(i)})'$ amplifies classically. Suppose we have an eigenstate $|\psi\rangle$ which violates exactly r constraints in layer i meaning $\langle\psi|H^{(i)}|\psi\rangle = r$. Dinur's theorem would then imply that on the transformed Hamiltonian

$$\langle\psi|(H^{(i)})'|\psi\rangle = \min(trc, cm) \tag{18}$$

where c is some constant and the second case applies if $r > \frac{m}{t}$. Suppose that $|\psi\rangle$ is a ground state of H with r violations. Then we can express

$$|\psi\rangle = \sum_{r=0}^{m} \alpha_r |\psi_r\rangle \tag{19}$$

Now lets look at the energy of $|\psi\rangle$ under $(H^{(i)})'$. This gives

$$\langle\psi|(H^{(i)})'|\psi\rangle = \sum_{r=0}^{m} \alpha_r^2 r' \tag{20}$$

If we now suppose that $\epsilon_0 = \lambda_{min}(H)$ is small then we get that

$$\begin{aligned}\langle\psi|(H^{(i)})'|\psi\rangle &\geq t\cdot c\left(\alpha_1^2 + 2\alpha_2^2 + \cdots + \frac{m}{t}\alpha_{m/t}^2\right) + c\cdot m\left(\alpha_{m/t+1}^2 + \cdots\right)\\ &\geq t\cdot c\left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2\right)\end{aligned} \tag{21}$$

where in the second step we get an upper bound by setting $t \ll m$. Here $\left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2\right)$ represents the weight with at least one violation. Now applying the detectability lemma we get

$$\langle\psi|(H^{(i)})'|\psi\rangle \geq t\cdot c\left(1 - \frac{1}{1+\epsilon_0/g^2}\right) \tag{22}$$

Now we go back to computing the ratio of the energy under the new vs. original Hamiltonian and we get

$$\frac{\langle\psi|(H^{(i)})'|\psi\rangle}{\langle\psi|H|\psi\rangle} \geq \frac{t\cdot c}{g^2} \tag{23}$$

Thus we've seen that if the ground energy is sufficiently small then we've amplified the energy by a factor tc/g^2. We would need a more sophisticated proof to make a statement when there are more violations.

References

[1] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani. *The Detectability Lemma and Quantum Gap Amplification.* STOC 2009 (extended version: arXiv:0811.3412).

[2] A. Anshu, I. Arad, and T. Vidick. *Simple proof of the detectability lemma and spectral gap amplification.* Phys. Rev. B 93, 205142 (2016). arXiv:1602.01210.