

## Lecture 15: Correlation Decay from the Detectability Lemma

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## 1 Decay of Correlations

We will begin looking at applications of the detectability lemma from last class. Today, we will focus on the exponential decay of correlations and consider area law in the next few lectures. We will

1. Define and motivate correlation functions (2-point).
2. Give examples.
3. Prove exponential decay in gapped frustration-free ground states.

**Definition 1.1.** The correlation of two observables  $A$  and  $B$  for a given quantum state  $|\psi\rangle$  is  $\mathcal{C}_{|\psi\rangle}(A, B) = \langle\psi|AB|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle$ . When  $A$  and  $B$  commute, this is the same as the covariance of two random variables when measuring in the common eigenbasis.



Figure 1: 1D quantum system

Examples:

- For  $|\psi_0\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ ,  $\mathcal{C}_{|\psi_0\rangle}(A, B) = 0$  when  $A$  and  $B$  only act on the qubits corresponding to  $|\psi_1\rangle$  and  $|\psi_2\rangle$  respectively.
- Since  $\mathcal{C}_{|\psi\rangle}(A, B) = \langle\psi|A|\psi\rangle \left[ \frac{\langle\psi|AB|\psi\rangle}{\langle\psi|A|\psi\rangle} - \langle\psi|B|\psi\rangle \right]$  when  $\langle\psi|A|\psi\rangle \neq 0$ , this can be thought of as the deviation of the expected value of  $B$  conditioned on  $A$  from its true expectation (scaled by the expectation of  $A$ ), which is clear in the classical case when  $A$  is a projection operator that commutes with  $B$ .
- CAT states and EPR pairs on the support of  $A$  and  $B$  have large correlation.

**Theorem 1.2** (Hastings [HK05]). *Given an  $O(1)$  local Hamiltonian with a unique ground state  $|\Omega\rangle$  of  $H$  and spectral gap  $\Delta$ , then  $\forall A, B$  with  $\|A\|, \|B\| \leq 1$ ,*

$$\mathcal{C}_{|\Omega\rangle}(A, B) \leq \tilde{O}(1)e^{-\Omega(\text{dist}(A, B)) \cdot \Delta}$$

where  $\text{dist}(A, B)$  is the length of the shortest path from  $\text{supp}(A)$  to  $\text{supp}(B)$  i.e. how many Hamiltonian terms are in the path in the interaction graph (edges included when contained in the support of a local Hamiltonian term). The  $\tilde{O}(1)$  term hides a minor dependence e.g. on the support of  $A$  and  $B$ .

We will focus on the gapped frustration free ground states case as it is simpler.

## 1.1 Intuition and Motivation

- In more specific models with explicit calculations, the decay of correlation was apparent, and so it is a signature of “locality” in the quantum state.
- For the toric code, far away operators are correlated, but the theorem doesn’t apply since there are four ground states.
- Whenever a Hamiltonian has short-range correlation, it is typically a sign that it is gapped (though not necessarily), and so correlation helps determine phase of matter. An example of a Hamiltonian that is gapless but has decay of correlation is the 1D Anderson Model.
- Decay of correlation in 1D suggests area law behavior and suggests using “numerical methods” like DMRG.

## 1.2 Examples

- AKLT model - any matrix product state with “injective/invertible maps” often have a parent Hamiltonian that is gapped.
- Transverse field Ising model:  $H = T \sum_i X_i + \sum_i Z_i Z_{i+1}$  for  $T \neq 1$  - has a gap and thus decay of correlation.
- $\sum_i |\theta_{i,i+1}\rangle \langle \theta_{i,i+1}|$  where  $|\theta_{i,i+1}\rangle = |01\rangle - \lambda |10\rangle$  for  $\lambda \neq 1$  - wants to move particles around but with the decay of  $\lambda$ .
- “Massive particles” - quantum mechanically described using quantum field theory which often lives in the low energy regime of local Hamiltonians where spectral gap scales with the gap. A classical example of this fact is as follows: consider the following spring system of a sequence of  $n$  particles with mass  $m$  connected sequentially through springs with constant  $k$ , with the leftmost particle connected to the wall with a force  $F(t)$  applied to the rightmost particle.

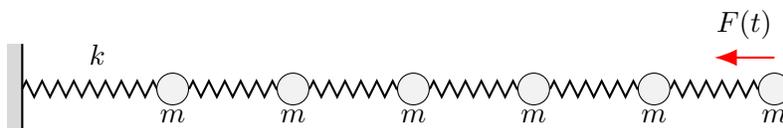


Figure 2: Classical example of “massive particles”

Let  $x_i$  be the position of particle  $i$  and so

$$m x_i'' = k(x_{i+1} - x_i - x_i + x_{i-1}) + 1_{i=n} F(t).$$

By noticing that the expression above is the discrete Laplacian, writing the equation in terms of the Fourier expansion, i.e. using

$$x_i = \int \hat{x}_i(\theta) e^{i\theta t}$$

one can solve the equation to get

$$|\hat{x}_i(\theta)| \sim e^{-c|i-n|}$$

when  $k < m$ , i.e. it can't pull the mass itself much. Since the gap corresponds to mass, we can see the decay of correlation exhibited through the bound above. Note that this isn't a perfect example since  $c$  in the above expression will be log of the mass whereas it should be mass in quantum field theory. Instead we would want the mass of the phonons and not the mass of the particles, so the discrepancy is coming from the different equations in quantum field theory.

## 2 Proof of Decay of Correlations

We will now prove the frustration-free version of the decay of correlations theorem, which will follow [AALV10]. This is still a useful result since later on it was modified in [GH15] to change the dependence on  $\Delta$  to instead be  $\sqrt{\Delta}$ , which is tight for frustration-free systems.

*Proof of Theorem 1.2.* Without much loss of generality, assume that  $H = \sum_i (\mathbb{1} - \Pi_{i,i+1})$  for projectors  $\Pi_{i,i+1}$  (change nonzero eigenvalues of local terms to 1, which we can do since it is often a constant e.g. if  $\|h_i\| = O(1)$ ). Recall the DL operator

$$\text{DL} = (\text{Prod}_{i \text{ even}} \Pi_{i,i+1}) (\text{Prod}_{i \text{ odd}} \Pi_{i,i+1})$$

which had the property that for  $|\psi\rangle$  orthogonal to the ground state  $|\Omega\rangle$ , we have

$$\|\text{DL} |\psi\rangle\|^2 \leq \frac{1}{1 + \Delta/4}.$$

Here, we will use the DL lemma to distinguish between ground and excited states, whereas in the last lecture, we used it to distinguish between frustration-free and frustrated Hamiltonians. Rewriting the above,

$$\langle \psi | \text{DL}^\dagger \text{DL} |\psi\rangle \leq \frac{1}{1 + \Delta/4} \leq 1 - \Delta/5$$

when  $\Delta \leq 1$ , whereas

$$\text{DL}^\dagger \text{DL} |\Omega\rangle = |\Omega\rangle$$

since by definition of being a frustration-free system,  $|\Omega\rangle$  satisfies all the local constraints. Thus,

$$(\text{DL}^\dagger \text{DL})^t |\psi\rangle \leq (1 - \Delta/5)^t \leq e^{-\Delta t/5}$$

i.e. DL is an approximate ground state projector (AGSP). Then

$$\mathcal{C}_{|\Omega\rangle}(A, B) = \langle \Omega | A (\mathbb{1} - |\Omega\rangle \langle \Omega|) B |\Omega\rangle = \langle \Omega | A (\mathbb{1} - (\text{DL}^\dagger \text{DL})^t) B |\Omega\rangle + \langle \Omega | A ((\text{DL}^\dagger \text{DL})^t - |\Omega\rangle \langle \Omega|) B |\Omega\rangle$$

where using the fact that  $(\text{DL}^\dagger \text{DL})^t$  is an AGSP, the second term is bounded by

$$\|A\| \|B\| e^{-\Delta t/5}.$$

To complete the proof, we will show that the first term is 0 when  $t \leq \text{dist}(A, B)/2 + 1$ , i.e.

$$\langle \Omega | A (\text{DL}^\dagger \text{DL})^t B |\Omega\rangle = \langle \Omega | AB |\Omega\rangle.$$

This can be seen from the following diagram:

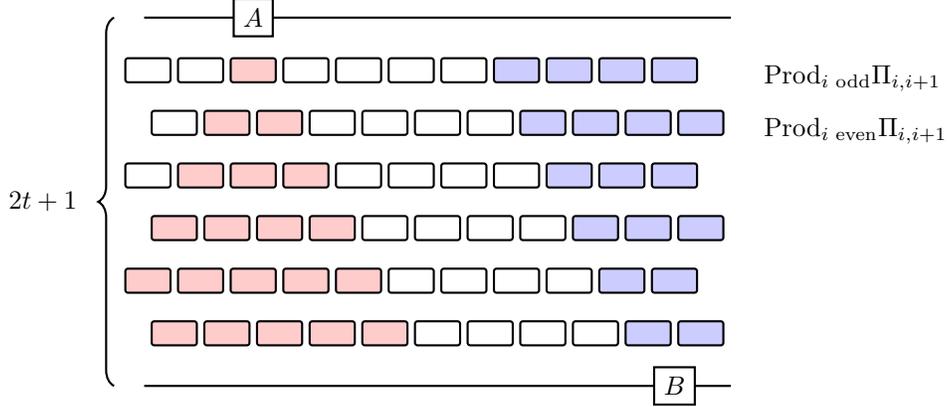


Figure 3: Lightcone argument for decay of correlations. Red and white operators can be absorbed downwards to  $|\Omega\rangle$  while blue and white operators can be absorbed upwards to  $\langle\Omega|$ .

We have  $(DL^\dagger DL)^t$  is equal to

$$(\text{Prod}_{i \text{ odd}} \Pi_{i,i+1}) [(\text{Prod}_{i \text{ even}} \Pi_{i,i+1}) (\text{Prod}_{i \text{ odd}} \Pi_{i,i+1})]^t$$

and so any operator that isn't within the lightcone associated to  $B$  can be commuted past it and disappear by applying it to  $|\Omega\rangle$ , which does nothing. Similarly, if the lightcone of  $B$  doesn't include  $A$ , then all the operators in the lightcone must also commute with  $A$ , and thus can be absorbed by applying it to  $\langle\Omega|$ , which happens when  $\text{dist}(A, B) \geq 2t + 2$  as each layer of projectors applied onto  $B$  can only decrease the distance by 1 at each step. Thus, it follows that

$$\langle\Omega| A (DL^\dagger DL)^t B |\Omega\rangle = \langle\Omega| AB |\Omega\rangle$$

so long as we choose  $t \leq \text{dist}(A, B)/2 - 1$ . □

### 3 Improvement by Gosset-Huang

Recall the Chebyshev polynomials can be defined as

$$K_l(\mathcal{O}) = \sum_{J=0}^l \alpha_J \mathcal{O}^J$$

for Hermitian operators  $\mathcal{O}$  which satisfies

$$K_l(1) = 1 \quad K_l(x) \leq 2e^{-l\sqrt{\epsilon}/2}$$

for  $0 \leq x \leq 1 - \epsilon$ . Chebyshev polynomials also have the property of being the smallest polynomial of a given degree in the region  $[0, 1 - \epsilon]$  when we assume that  $K_l(1) = 1$ . Gosset and Huang were able to improve the dependence on  $\Delta$  to be  $\sqrt{\Delta}$  by replacing  $(DL^\dagger DL)^t$  with  $K_t(DL^\dagger DL)$ . Then in the above argument, what changes are the following:

- We now have

$$\langle\Omega| A (K_t(DL^\dagger DL) - |\Omega\rangle\langle\Omega|) B |\Omega\rangle \leq 2\|A\|\|B\|e^{-\Omega(t\sqrt{\Delta})}$$

- The first term is now

$$\langle \Omega | A(\mathbb{1} - K_t(\text{DL}^\dagger \text{DL}))B | \Omega \rangle$$

and since  $K_t$  is as polynomial of degree at most  $t$ , by linearity and the previous lightcone argument, we again get that this is 0.

Thus the Gosset-Huang improvement is a direct consequence of just replacing everything by the Chebyshev polynomial.

## 4 Beyond Frustration-Free Systems

We now consider the more general case with possibly frustrated systems. It turns out the decay of correlations holds as long as the conditions for the Lieb-Robinson bound holds as in the case of bounded strength geometrically local Hamiltonians. The version we will use is the following:

**Lemma 4.1** (Lieb-Robinson Bound [Has21]). *For a Hamiltonian  $H = \sum_X h_X$ , suppose there are constants  $\mu, s > 0$  such that all sites  $i$  satisfy*

$$\sum_{X \ni i} \|h_X\| |X| e^{\mu \cdot \text{diam}(X)} \leq s < \infty.$$

Define the Lieb-Robinson velocity to be  $v_{LR} = 4s/\mu$ . Then for operators  $A, B$  with  $\text{dist}(A, B) \geq v_{LR}t$ ,

$$\|[A(t), B]\| \leq |X| \|A\| \|B\| e^{-\frac{\mu}{2} \text{dist}(A, B)}$$

where  $A(t) = e^{iHt} A e^{-iHt}$ .

The case we consider will be when  $\mu, s, v_{LR} = \Theta(1)$  as in the setting for theorem 1.2. First, center  $A$  and  $B$  to be  $A - \langle \Omega | A | \Omega \rangle I$  and  $B - \langle \Omega | B | \Omega \rangle I$  respectively, which only scales the norm of  $A$  and  $B$  by at most 2, so it suffices to prove decay of correlations on these centered operators instead and bound  $\langle \Omega | AB | \Omega \rangle$ . Consider a spectral decomposition of  $H = \sum_{n \geq 0} E_n |\psi_n\rangle \langle \psi_n|$  where  $|\Omega\rangle = |\psi_0\rangle$  and  $E_0 < E_1 \leq E_2 \leq \dots$  so that  $e^{iHt} = \sum_{n \geq 0} e^{iE_n t} |\psi_n\rangle \langle \psi_n|$ . Then expanding  $\langle \psi_0 | [A(t), B] | \psi_0 \rangle$  gives

$$= e^{iE_0 t} \langle \psi_0 | A \sum_{n \geq 0} e^{-iE_n t} |\psi_n\rangle \langle \psi_n| B | \psi_0 \rangle - \langle \psi_0 | B \sum_{n \geq 0} e^{iE_n t} |\psi_n\rangle \langle \psi_n| A | \psi_0 \rangle e^{-iE_0 t} \quad (1)$$

$$= \sum_{n \geq 0} \langle \psi_0 | A | \psi_n \rangle \langle \psi_n | B | \psi_0 \rangle e^{-i(E_n - E_0)t} - \sum_{n \geq 0} \langle \psi_0 | A | \psi_n \rangle \langle \psi_n | B | \psi_0 \rangle e^{i(E_n - E_0)t} \quad (2)$$

To get  $\langle \psi_0 | AB | \psi_0 \rangle$ , it suffices to get the first sum or “negative frequency” part when  $t = 0$  in which case they are equal. One way of extracting this is by choosing a kernel  $K(t)$  so that

$$\int K(t) e^{-iEt} dt \approx \begin{cases} 1 & E \geq \Delta \\ 0 & E \leq -\Delta \end{cases}$$

since then we can integrate 2 against  $K(t)$  annihilating the second sum while preserving the first. This can be done approximately via the following kernel:

**Lemma 4.2** ([Has21]). For  $E \in \mathbb{R}$  and  $\alpha > 0$ ,

$$\lim_{T \uparrow \infty} \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{-T}^T \frac{e^{-\alpha t^2}}{t + i\epsilon} e^{-iEt} dt = \begin{cases} 1 + O(\exp(-\Delta^2/4\alpha)) & E \geq \Delta \\ O(\exp(-\Delta^2/4\alpha)) & E \leq -\Delta \end{cases}$$

Applying this to 2 and rearranging gives

$$\langle \psi_0 | AB | \psi_0 \rangle = \lim_{T \uparrow \infty} \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{-T}^T \frac{e^{-\alpha t^2}}{t + i\epsilon} \langle \psi_0 | [A(t), B] | \psi_0 \rangle dt + O(\exp(-\Delta^2/4\alpha)).$$

Now choose  $\alpha = \Delta v_{LR} / \text{dist}(A, B)$  so the second term is at most  $e^{-\Omega(\text{dist}(A, B)) \cdot \Delta}$ . The first term can be bounded by splitting the integral for  $|t| \leq \text{dist}(A, B) / v_{LR}$  and  $|t| \geq \text{dist}(A, B) / v_{LR}$ . For the first case, we can apply the Lieb-Robinson bound giving

$$\lesssim \|X\| \|A\| \|B\| e^{-\Omega(\text{dist}(A, B))}$$

and the second case can be bounded using the triangle inequality, which gives

$$\lesssim \|A\| \|B\| \lim_{T \uparrow \infty} \lim_{\epsilon \downarrow 0} \int_{|t| \geq \text{dist}(A, B) / v_{LR}} \frac{e^{-\alpha t^2}}{t + i\epsilon} dt \lesssim \|A\| \|B\| e^{-\Omega(\alpha \cdot \text{dist}(A, B)^2)} \lesssim \|A\| \|B\| e^{-\Omega(\text{dist}(A, B)) \cdot \Delta}.$$

## References

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- [HK05] Matthew Hastings and Tohru Koma. Spectral gap and exponential decay of correlations. *Communications in Mathematical Physics*, 265, 07 2005.