Hamiltonian Decoded Quantum Interferometry

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Optimization by decoded quantum interferometry

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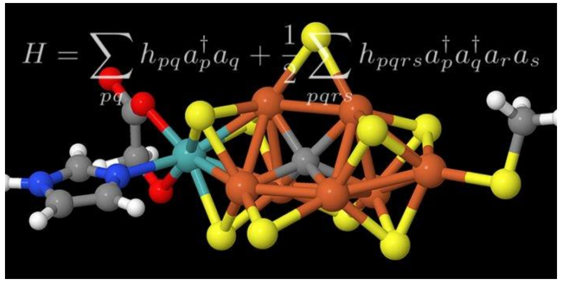
Hamiltonian Decoded Quantum Interferometry

Alexander Schmidhuber[*1,2], Jonathan Z. Lu[*2], Noah Shutty[1], Stephen Jordan[1], Alexander Poremba[†2,3], and Yihui Quek[†2,4]

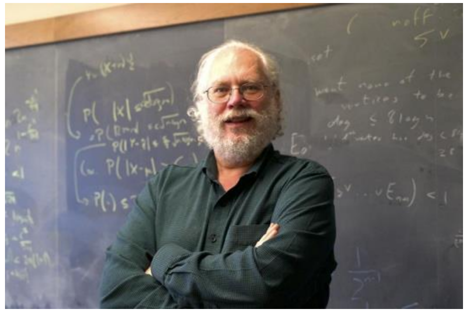
What are quantum computers good at?



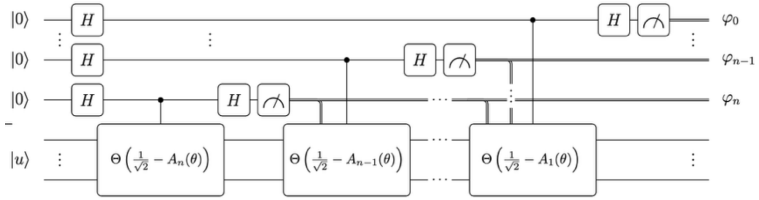
Quantum chemistry / Material science

Exponential quantum speedups?

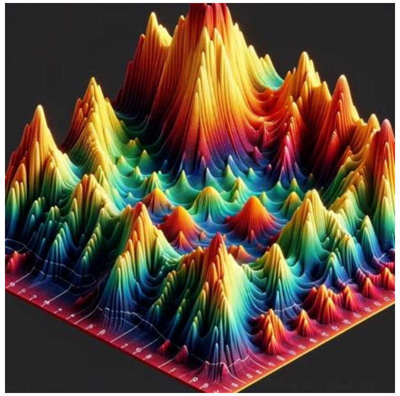
via classical decoding!



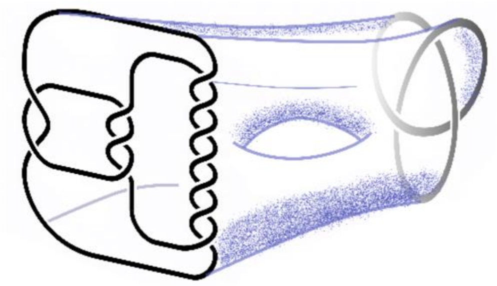
Integer factorization / HSP



Matrix inversion / Linear algebra



Combinatorial optimization?



Approximating topological invariants

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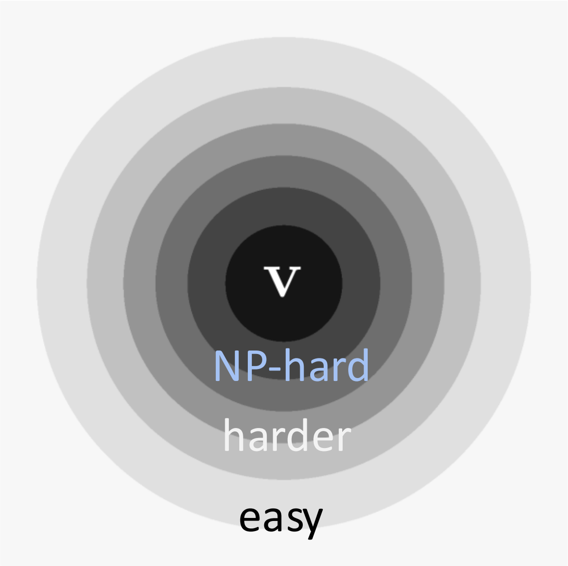
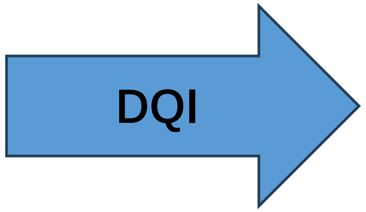
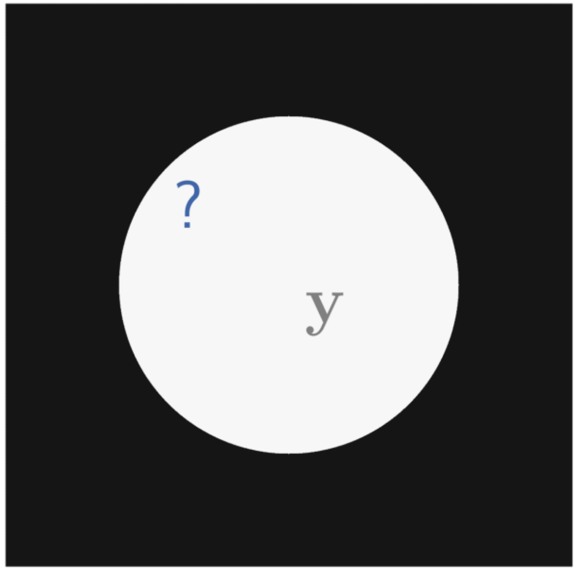
Alexander Schmidhuber, MIT

DQI is a quantum reduction from classical optimization to classical decoding



$\boxed{\max_{\mathbf{x}} f(\mathbf{x})}$





Decode a noisy message

quantum algorithm for optimization = quantum computer + classical decoder

Example

Given:

m* constraints on *n* variables over a binary field mod *2

$$B\mathbf{x} \equiv \mathbf{v}$$

$$B = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} \in \mathbb{F}_2^{m\times n} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{F}_2^m$$

Goal:

"satisfy as many $\mathbf{b}_i \cdot \mathbf{x} = v_i$ as possible"

$=$

maximize

$$f(\mathbf{x}) = \sum_{j=1}^{m} (-1)^{\mathbf{b}_j \cdot \mathbf{x} + v_j} = \#\mathrm{SAT} - \#\mathrm{UNSAT}$$

This defines a code: $C^\perp = \{\mathbf{d} \in \mathbb{F}_2^m : B^T\mathbf{d} = \mathbf{0}\}$

and a decoding problem: Given $\boldsymbol{s} = B^T\boldsymbol{e}$, recover $\boldsymbol{e}$ provided $|\boldsymbol{e}| \le \ell$

Central claim of DQI:

Decode up to ℓ errors on $C^\perp \implies$ Sample $\mathbf{x} \sim P(f(x))^2$,

where P is an arbitrary degree-ℓ polynomial

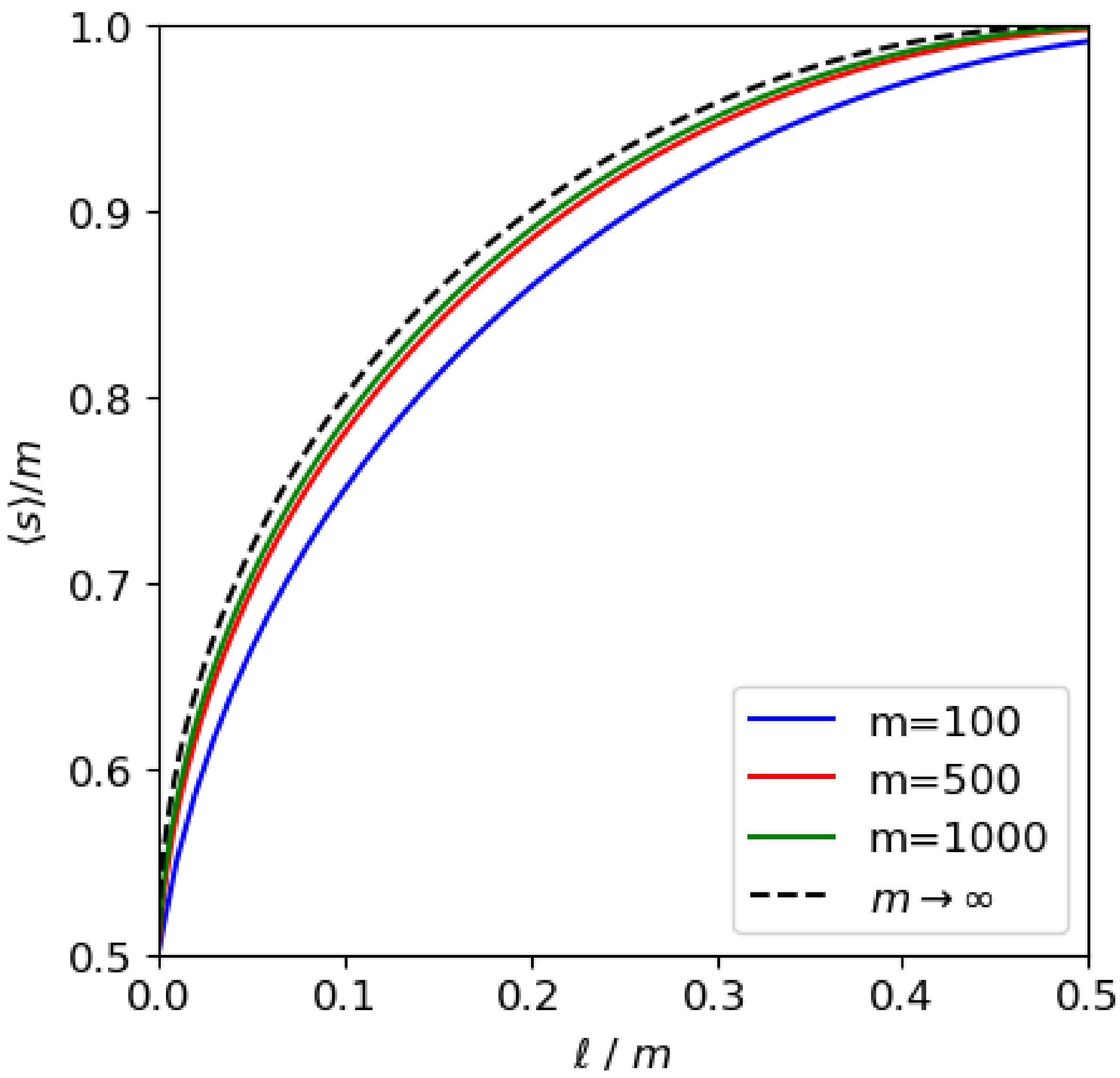
Performance of DQI

- How can we predict the performance of DQI?

$$B\mathbf{x} \equiv \mathbf{v}$$

If one can correct weight-ℓ errors on $C^{\perp}$, then DQI find a solution **x** that satisfies **s** constraints given by

$$\frac{\langle s \rangle}{m} = \frac{1}{2} + \sqrt{\frac{\ell}{m}\left(1 - \frac{\ell}{m}\right)}$$



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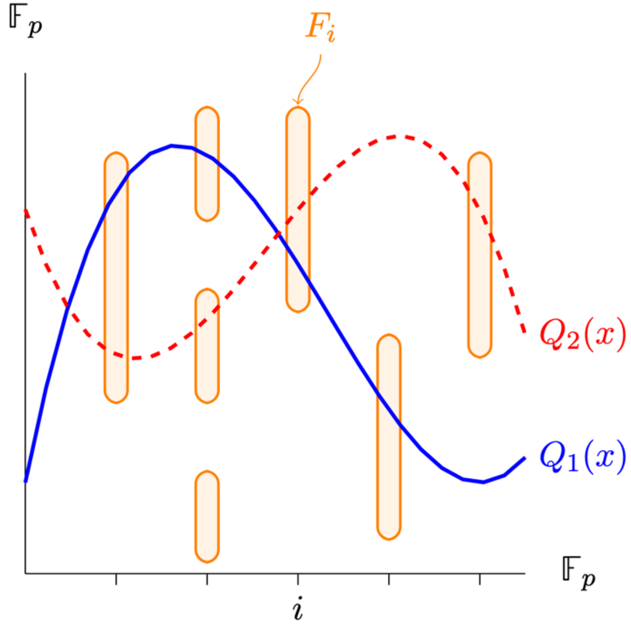
What kind of structure benefits DQI?

algebraic structure

Optimization problem: "Optimal Polynomial Intersection"

DQI

Decoding problem: Reed-Solomon code

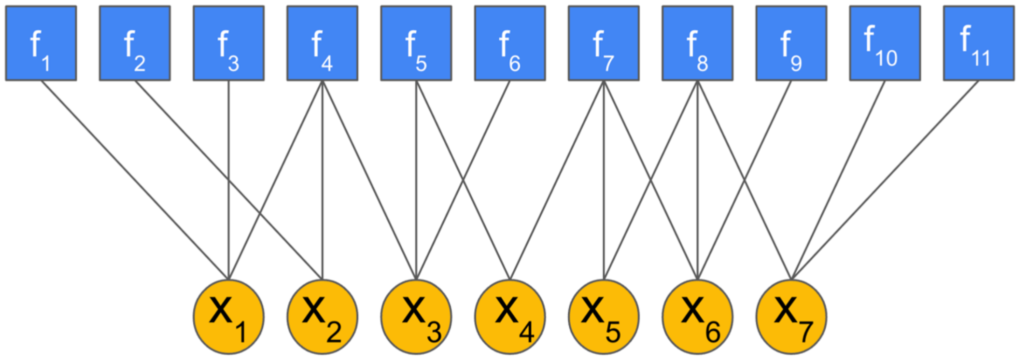


sparsity

Optimization problem: Sparse CSP

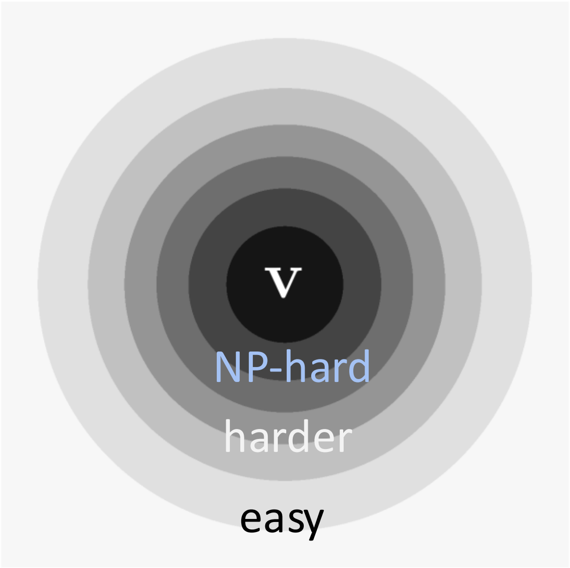
DQI

Decoding problem: LDPC code

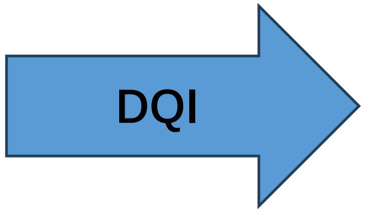


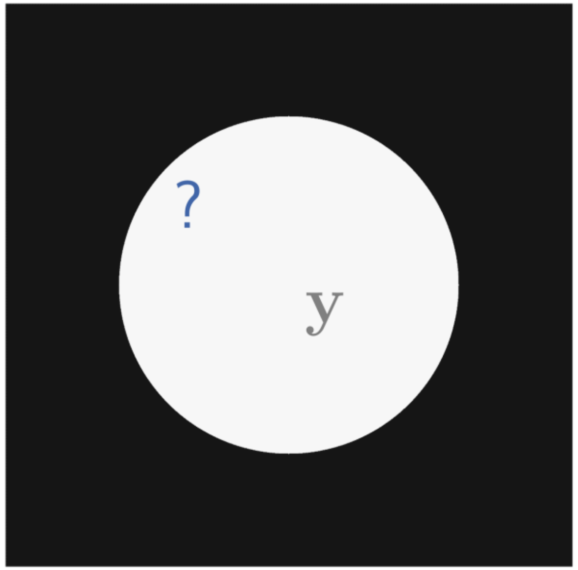
2

Hamiltonian DQI



$\boxed{\max_{\mathbf{x}} f(\mathbf{x})}$

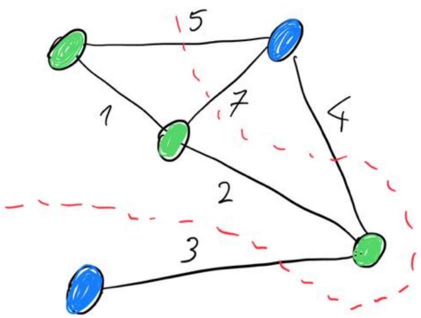




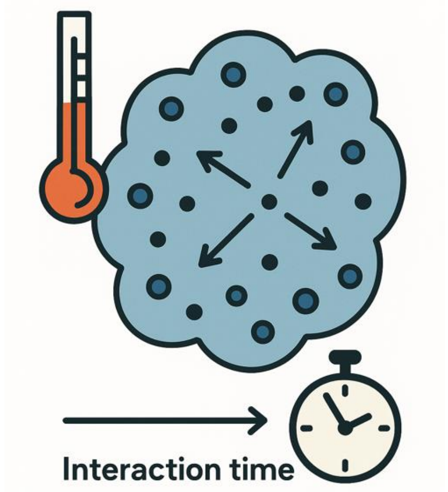
Decode a noisy message

quantum algorithm for optimization = quantum computer + classical decoder

Hamiltonian Optimization



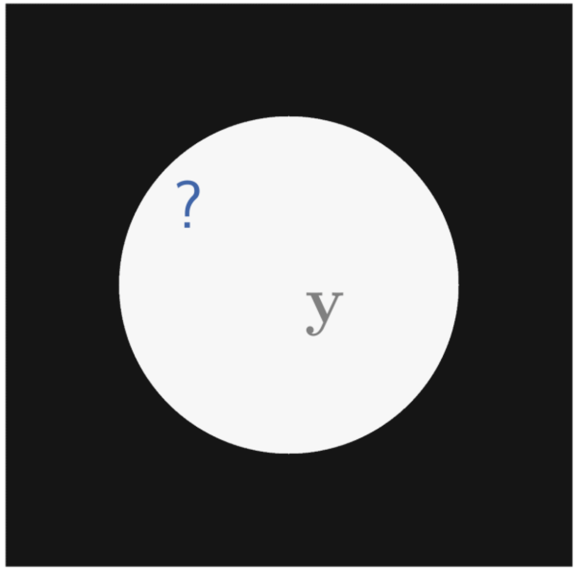
Gibbs sampling



Hamiltonian simulation

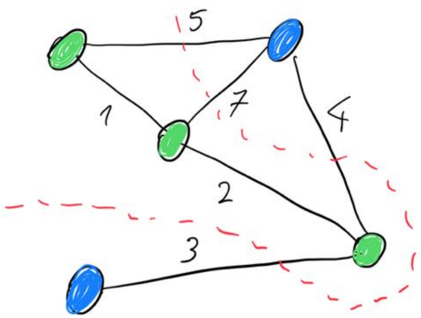
$$e^{-iHt}$$

?

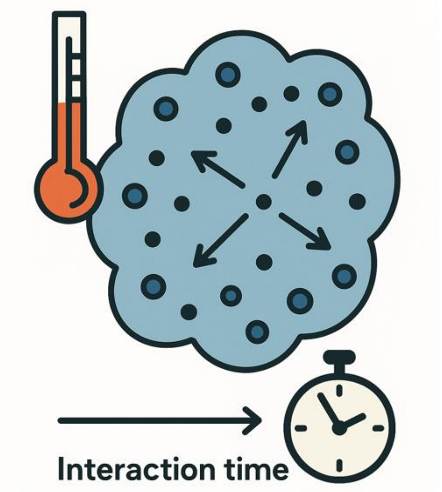


Decode a noisy message

Hamiltonian Optimization



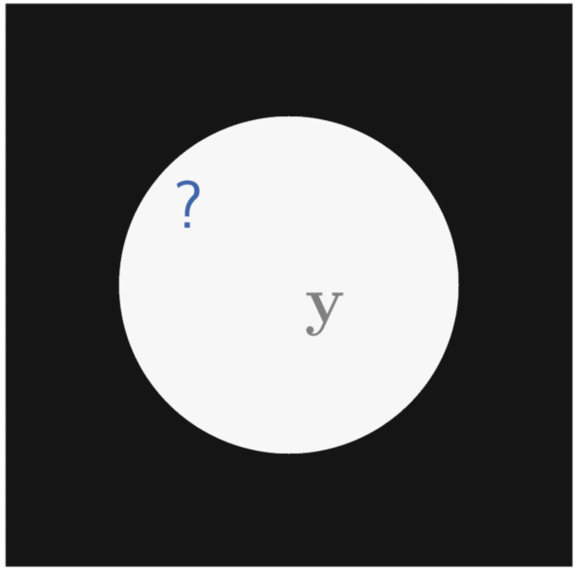
Gibbs sampling



Hamiltonian simulation

$$e^{-iHt}$$

Hamiltonian DQI



Decode a noisy message

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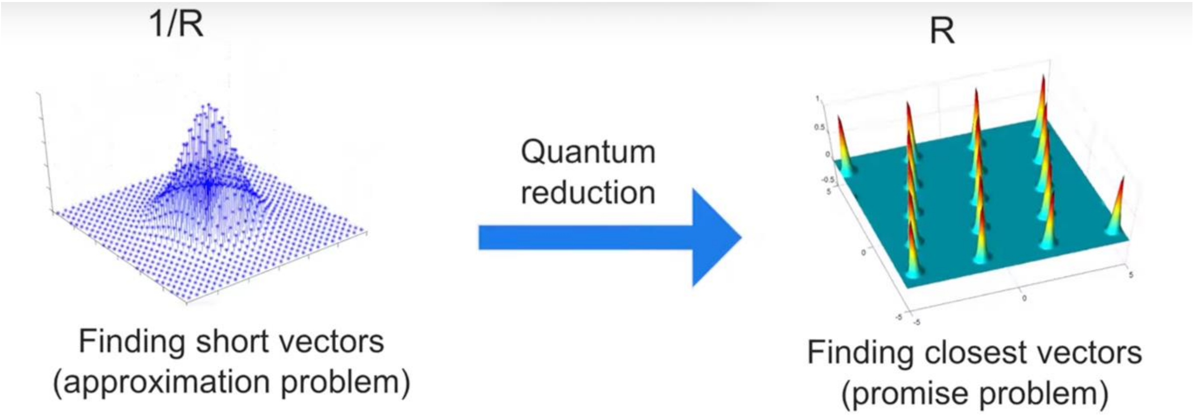
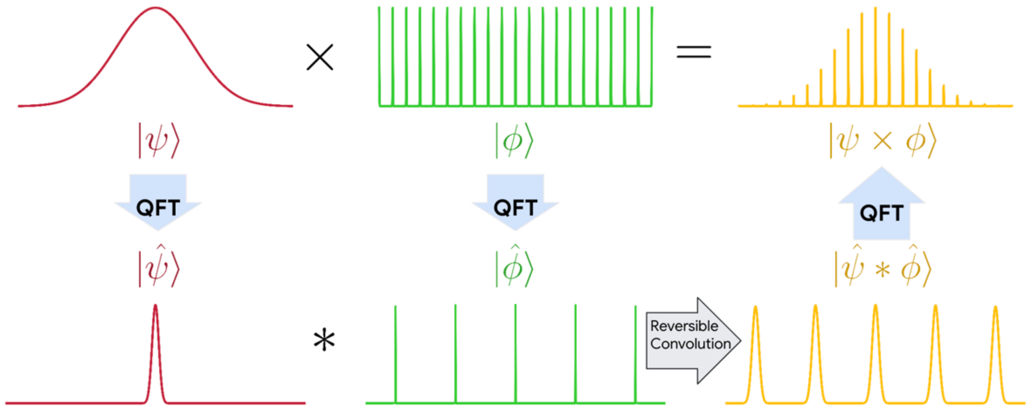
Alternative: Non-abelian Regev's reduction



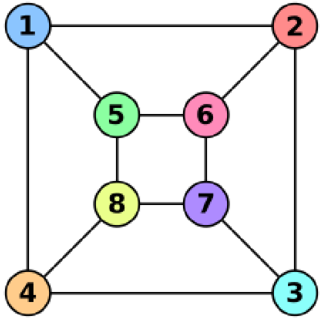


Abelian groups:

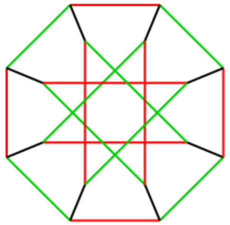
$$\mathbb{F}_2, \mathbb{R}, V$$

Non-abelian groups:

$$\begin{array}{c} i<j \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix} \\ \pi(i)>\pi(j) \end{array}$$



$$X = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$



Hamiltonian optimization

Given:

m* constraints on *n* variables over a binary field mod *2

$$B = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} \in \mathbb{F}_2^{m \times n} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{F}_2^m$$

$$H = \sum_i (-1)^{v_i} Z^{b_i} \qquad H|x\rangle = f(x)|x\rangle \qquad f(\mathbf{x}) = \sum_{j=1}^{m} (-1)^{\mathbf{b}_j \cdot \mathbf{x} + v_j} = \#\text{SAT - }\#\text{UNSAT}$$

Task: Sample $x \sim P(f(x))^2$

(Pauli) Hamiltonian generalization:

Given:

$$H = \sum_{i=1}^{m} v_i P_i \qquad H\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

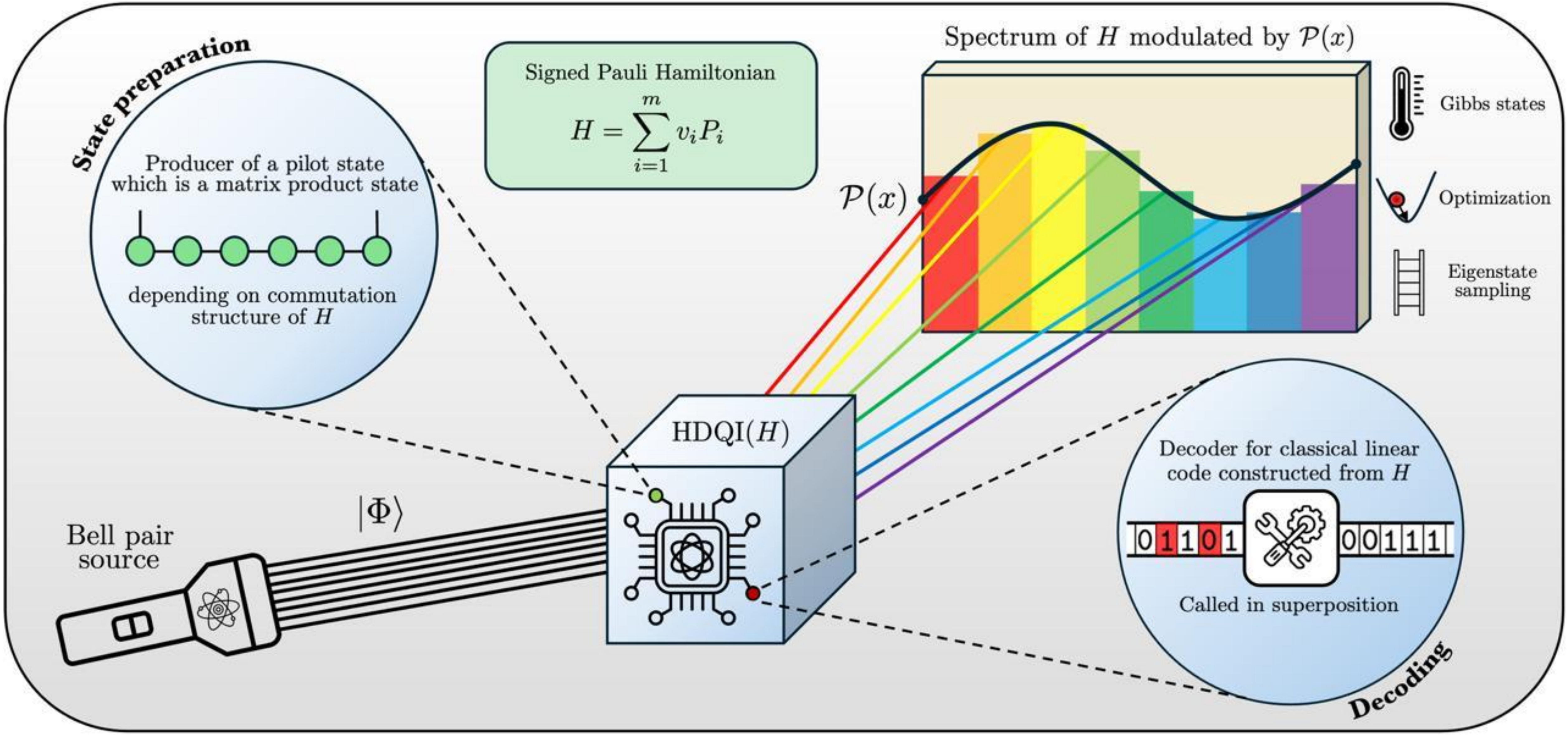
Task: Sample $|\lambda\rangle \sim \mathcal{P}(\lambda)^2$

$$\rho_{\mathcal{P}}(H) = \frac{\mathcal{P}^2(H)}{\mathrm{Tr}\left[\mathcal{P}^2(H)\right]}$$

HDQI reduces this task to:

Decoding + Preparing a Pilot state

Alexander Schmidhuber, MIT



Algorithm

Goal: Sample $|\lambda\rangle \sim \mathcal{P}(\lambda)^2$.

DQI prepares

$$\propto \sum_{\mathbf{x}\in\mathbb{F}_2^n} \mathcal{P}(f(\mathbf{x}))|\mathbf{x}\rangle$$

Naïve approach for HDQI:

$$\propto \mathcal{P}(H)\sum_\lambda |\lambda\rangle = \sum_\lambda \mathcal{P}(\lambda)\,|\lambda\rangle$$

Two issues

Resolution:

$$|\Phi^n\rangle = \frac{1}{2^{n/2}} \sum\nolimits_{x\in\mathbb{F}_2^n} |x\rangle \otimes |x\rangle \; = \frac{1}{2^{n/2}} \sum_\lambda |\lambda\rangle \otimes |\bar{\lambda}\rangle$$

$$(\mathcal{P}(H)\otimes \mathbb{1})\,|\Phi^n\rangle = \frac{1}{2^{n/2}} \sum_\lambda \mathcal{P}(\lambda)\,|\lambda\rangle \otimes |\bar{\lambda}\rangle \xrightarrow{\text{Partial trace}} \rho_{\mathcal{P}}(H) = \frac{\mathcal{P}^2(H)}{\mathrm{Tr}\left[\mathcal{P}^2(H)\right]}$$

Algorithm (commuting case)

Input:

$$H = \sum_{i=1}^{m} v_i P_i$$

Goal: Prepare

$$(\mathcal{P}(H) \otimes \mathbb{1}) |\Psi\rangle := (\mathcal{P}(H) \otimes \mathbb{1}) \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} |x\rangle \otimes |x\rangle$$

1. Expand

$$\mathcal{P}(H) = \sum_{k=0}^{\ell} w_k \sum_{y \in \mathbb{F}_2^m, |y|=k} \left(\prod_{i \in y} v_i \right) P_y, \qquad P_y := \prod_{i \in y} P_i,$$

2. Prepare pilot state

$$\sum_{k=0}^{\ell} w_k \sum_{y \in \mathbb{F}_2^m, |y|=k} |y\rangle |\Phi^n\rangle ,$$

3. Controlled Paulis

$$\sum_{k=0}^{\ell} w_k \cancel{\sum_{y \in \mathbb{F}_2^m, |y|=k}} |y\rangle \left(\prod_{i \in y} v_i \right) P_y |\Phi^n\rangle .$$

4. Uncompute!

$$|y\rangle P_y |\Phi^n\rangle \mapsto |0^m\rangle P_y |\Phi^n\rangle$$

Decoding problem:

Recover $y \in \mathbb{F}_2^m$ (with $|y| \leq \ell$) given $P_y \sum_x |x\rangle|x\rangle$

$$P_y := \prod_{i \in y} P_i$$

Step 1: Recover P_y from $P_y \sum_x |x\rangle|x\rangle$

Information theoretically possible: $\forall P, Q \in \{X, Y, Z, \mathbb{1}\}^{\otimes n} : \langle\Psi| PQ |\Psi\rangle = \frac{1}{2^n} \mathrm{tr}\{PQ\} = \delta_{P,Q}$.

Efficiently possible (Bell measurement): $|\Psi\rangle = \left(|\Phi^+\rangle_{1,1'}\right) \otimes \left(|\Phi^+\rangle_{2,2'}\right) \otimes \cdots \otimes \left(|\Phi^+\rangle_{n,n'}\right)$

Step 2: Recover $y \in \mathbb{F}_2^m$ (with $|y| \leq \ell$) from P_y

Classical decoding problem

Decoding problem

$$G_n \equiv \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}^{\otimes n}$$

$$P_{\mathbf{a}} = i^{\mathbf{a}^T Y \mathbf{a}} \prod_{j=1}^{n} X_j^{\mathbf{a}_{2j-1}} Z_j^{\mathbf{a}_{2j}}, \; a = \mathrm{symp}(\mathrm{P_i}) \in \mathbb{F}_2^{2\mathrm{n}}$$

<u>Group homomorphism</u>

$$\mathrm{symp}(P_i P_j) = \mathrm{symp}(P_i) \oplus \mathrm{symp}(P_j) \in \mathbb{F}_2^{2n}$$

Bell measurement maps $P_y \sum_x |x\rangle|x\rangle$ **to** $\mathrm{symp}(P_y)$

Decoding problem:

Recover y from $P_y \sum_x |x\rangle|x\rangle$ $\qquad P_y := \prod_{i \in y} P_i$

Step 2: Recover $y \in \mathbb{F}_2^m$ (with $|y| \leq \ell$) from $\mathrm{symp}(P_y)$

This is just a classical bounded-distance syndrome decoding problem

$$B^T y \mapsto y \text{ given } |y| \leq \ell \qquad B \in \mathbb{F}_2^{m \times 2n}$$

Symplectic code defined by parity check matrix

$$B^T = \begin{bmatrix} | & | & & | \\ \mathrm{symp}(P_1) & \mathrm{symp}(P_2) & \cdots & \mathrm{symp}(P_m) \\ | & | & & | \end{bmatrix} \in \mathbb{F}_2^{2n \times m}$$

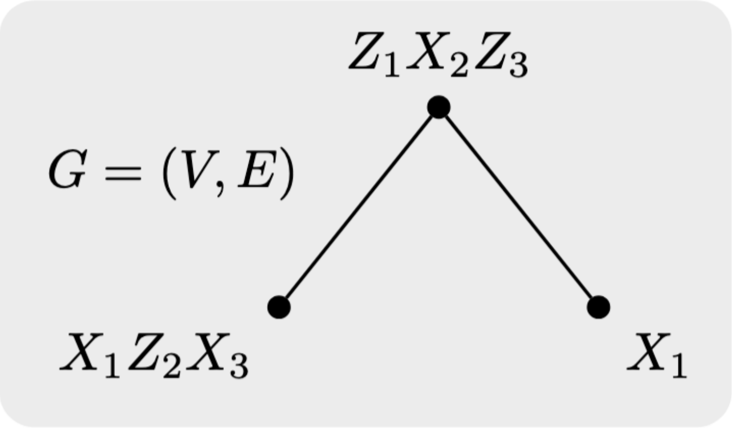
Non-commuting case:

There are now cancellations!

$$\mathcal{P}(H) = \sum_{k=0}^{\ell} w_k \sum_{y \in \mathbb{F}_2^m, |y|=k} \alpha(y) P_y,$$

$$\alpha(y) = \frac{1}{|y|!} \sum_{\sigma \in S^{|y|}} \mathrm{sgn}_y(\sigma) \qquad \mathrm{sgn}_y(\sigma) := \frac{\sigma(P_y)}{P_y}$$

$$H = X_1 Z_2 X_3 + Z_1 X_2 Z_3 - X_1$$



New goal:
Prepare the pilot state

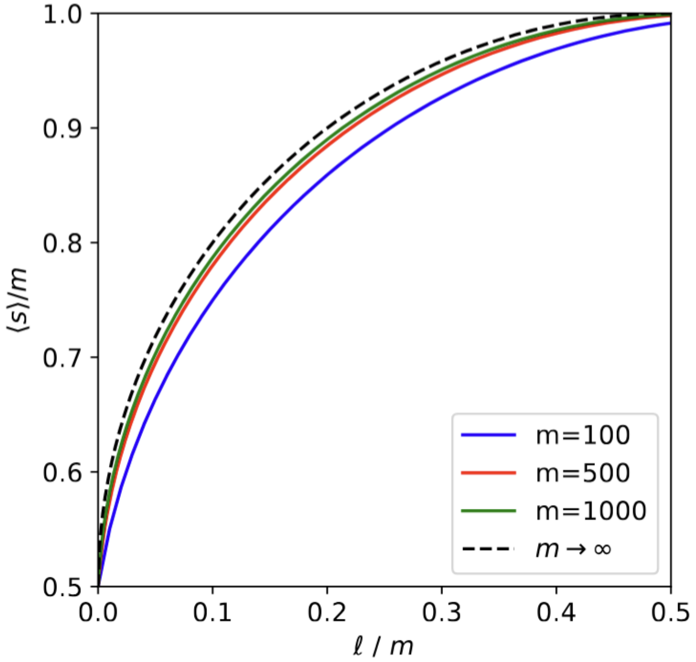
$$\sum_{k=0}^{\ell} w_k \sum_{y \in \mathbb{F}_2^m, |y|=k} \alpha(y) \left|y\right\rangle \left|\Psi\right\rangle$$

3

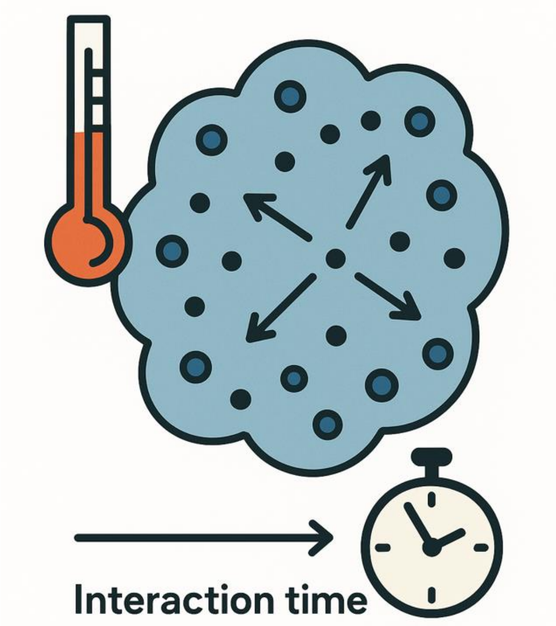
Applications of HDQI

Applications:

Optimization and Gibbs sampling



$$\frac{\langle E\rangle}{m} = \left(\sqrt{\frac{\ell}{2m}} + \sqrt{\left(\frac{1}{2} - \frac{\ell}{2m}\right)}\right)^2$$



$$\beta||H|| = \frac{1}{\sqrt{2}}\ell$$

Applications

	sparsity	algebraic structure
commuting		
non-commuting		

Random commuting local Hamiltonians

$$H = \sum_i v_i P_i \qquad [P_i, P_j] = 0 \qquad P_j \in \mathcal{P}_n^{(k)} := \{k\text{–local Paulis on } n \text{ qubits}\}$$

$$B^T = \begin{bmatrix} | & | & & | \\ \mathrm{symp}(P_1) & \mathrm{symp}(P_2) & \cdots & \mathrm{symp}(P_m) \\ | & | & & | \end{bmatrix}$$

Local Hamiltonian → LDPC code!

$(m/n, k)$	HDQI and BP	Clifford SA
$(3,3)$	$69.15 \pm 0.04\%$	$77.92 \pm 0.29\%$
$(3,4)$	$69.73 \pm 0.04\%$	$76.87 \pm 0.32\%$
$(3,5)$	$69.25 \pm 0.03\%$	$75.68 \pm 0.30\%$
$(3,6)$	$68.57 \pm 0.03\%$	$74.60 \pm 0.51\%$
$(6,3)$	$61.56 \pm 0.04\%$	$70.10 \pm 0.26\%$
$(6,4)$	$62.38 \pm 0.02\%$	$69.35 \pm 0.29\%$
$(6,5)$	$62.31 \pm 0.02\%$	$68.56 \pm 0.28\%$
$(6,6)$	$62.00 \pm 0.02\%$	$67.46 \pm 0.21\%$
$(10,3)$	$58.02 \pm 0.03\%$	$65.44 \pm 0.31\%$
$(10,4)$	$58.83 \pm 0.02\%$	$64.81 \pm 0.18\%$
$(10,5)$	$58.90 \pm 0.02\%$	$64.31 \pm 0.27\%$
$(10,6)$	$58.75 \pm 0.01\%$	$63.60 \pm 0.32\%$

Fig: Energy (in percentages) achieved by HDQI and a classical Clifford SA algorithm.

Structured commuting Hamiltonians

Theorem:

if $\dim \mathrm{Ker} B^T = \text{const}$

$$B^T = \begin{bmatrix} | & | & & | \\ \mathrm{symp}(P_1) & \mathrm{symp}(P_2) & \cdots & \mathrm{symp}(P_m) \\ | & | & & | \end{bmatrix}$$

$\Rightarrow$

HDQI efficiently prepares

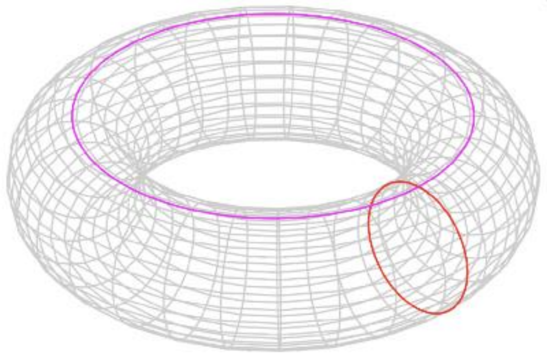
$$\rho_H = f(H)/\,\mathrm{Tr}[f(H)]$$

for any non-zero function $f : \mathrm{Spec}(H) \to \mathbb{R}_{\geq 0}$

Example: Toric code

$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p$$

$$A_v = \prod_{i \in v} X_i, \qquad B_p = \prod_{i \in p} Z_i$$



Two relations:

$$\prod_{v \in V} A_v = \prod_{p \in F} B_p = \mathbb{1}.$$

$$\dim \mathrm{Ker} B^T = 2^2 = 4$$

HDQI prepares arbitrary Gibbs states of the 2D Toric code

Polynomial-Time Preparation of Low-Temperature Gibbs States for 2D Toric Code

Zhiyan Ding[*1], Zeph Landau[†2], Bowen Li[‡3], Lin Lin[§1,4], and Ruizhe Zhang[¶5]

Gibbs state preparation for commuting Hamiltonian: Mapping to classical Gibbs sampling

Yeongwoo Hwang[‡ *1] and Jiaqing Jiang[‡ †2]

Efficient and simple Gibbs state preparation of the 2D toric code via duality to classical Ising chains

Pablo Páez Velasco,[1, 2, *] Niclas Schilling,[3, †] Samuel O. Scalet,[4, 5, ‡] Frank Verstraete,[4, 6, §] and Ángela Capel[4, 7, ¶]

Alexander Schmidhuber, MIT

Structured commuting Hamiltonians

Theorem:

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$$B^T = \begin{bmatrix} | & | & & | \\ \mathrm{symp}(P_1) & \mathrm{symp}(P_2) & \cdots & \mathrm{symp}(P_m) \\ | & | & & | \end{bmatrix}$$

$\Rightarrow$

HDQI efficiently prepares

$$\rho_H = f(H)/\,\mathrm{Tr}[f(H)]$$

for any non-zero function $f : \mathrm{Spec}(H) \to \mathbb{R}_{\geq 0}$

Model	**Geometry**	**$\dim \mathrm{Symp}(H)$**	**Reason / remarks**
Ising (ring)	1D, periodic	1	Unique cycle relation $\prod ZZ = I$.
Surface/toric	2D, closed	2	Global star and plaquette products; homology on closed surfaces.
Color code (stabilizer)	2D, closed	4	Two independent two-color products per Pauli sector.
TI Pauli in 2D	2D, closed	$2K$	LC-equivalent to K toric copies; K constant.
Cluster (1D, 2D, 3D)	periodic	0	Unique stabilizer ground state.
Finite stack of 2D codes	embedded in 3D	$2K$	Sum over K decoupled layers.
Haah (generic sizes)	3D, periodic	$O(1)$	No local relations; global count constant for many L.
		Counterexamples	
3D toric code	3D, periodic	$\Theta(L^3)$	Cube-local plaquette identities.
X-cube / checkerboard	3D, periodic	$\Theta(L^3)$	Many local relations (type-I fracton).

Noah Shutty

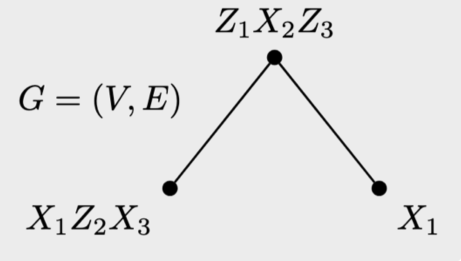
``This is also classically easy!”

Alexander Schmidhuber, MIT

Non-commuting Hamiltonians

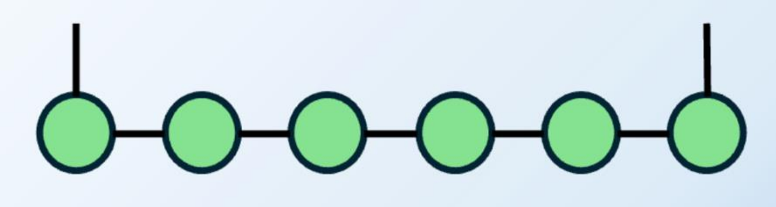
Theorem:

If the anti-commutation graph factorizes into connected components of size $O(\log(n))$



⇒

The pilot state can be prepared efficiently as a Matrix Product State (MPS).



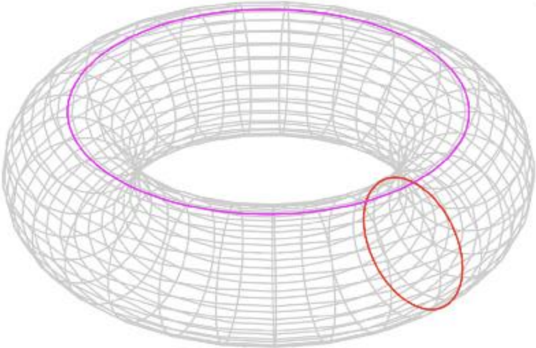
Semi-classical spin glass:

$$H(p) = \sum_{i=1}^{m} v_i P(\mathbf{b}_i)$$

$$P(\mathbf{b}_i) = \begin{cases} Z^{\mathbf{b}_i} \text{ with probability } 1-p \\ \mathrm{X}^{\mathbf{b}_i} \text{ with probability p} \end{cases}$$

Commuting Hamiltonians with random defects:

$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p$$



The Pilot state as a QMA witness

- Consider a Hamiltonian H for which the decoding problem is easy.
- Then the Pilot state acts as a witness for certifying the ground state energy.

$$\sum_{k=0}^{\ell} w_k \sum_{y \in \mathbb{F}_2^m, |y|=k} \alpha(y) |y\rangle |\Psi\rangle \qquad \alpha(y) = \frac{1}{|y|!} \sum_{\sigma \in S^{|y|}} \mathrm{sgn}_y(\sigma) \qquad \mathrm{sgn}_y(\sigma) := \frac{\sigma(P_y)}{P_y}$$

- Given this state, there is an efficient quantum algorithm (not QPE!) that computes the ground state energy.
- The Pilot state only depends on the anti-commutation graph (not on, e.g., signs!)
- QMA with advice?

Future directions

- **When can the pilot state be efficiently prepared?**
  - Beyond log(n)-sized connected components?
  - Exploit structure beyond sparsity?
- **What interesting quantum Hamiltonians correspond to good classical codes?**
  - Random local Hamiltonians
  - Topological code Hamiltonians
  - …?
- **Generalizations to other non-abelian groups?**

Thank you

Alternative view of HDQI

Decoded Quantum Interferometry:

QFT over the abelian group $\mathbb{F}_2^n$

$$H^{\otimes n}$$

$$
\begin{aligned}
|0^m\rangle|0^n\rangle &\xrightarrow{\text{State prep}} \sum_{e\in\mathbb{F}_2^m} |e\rangle|0^n\rangle \\
&\xrightarrow{\text{Apply } X^{B^T e}} \sum_{e\in\mathbb{F}_2^m} |e\rangle|B^T e\rangle \\
&\xrightarrow{\text{Decoder}} \sum_{e\in\mathbb{F}_2^m} |0\rangle|B^T e\rangle \\
&\xrightarrow{\text{Fourier transform}} \sum_{x\in\mathbb{F}_2^n} \mathcal{P}(f(x))|x\rangle
\end{aligned}
$$

Hamiltonian DQI:

QFT over non-abelian group $\{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$

$$V = (H^{\otimes n} \otimes \mathbb{1})\ \mathrm{CNOT}^{\otimes n}$$

Bell state preparation unitary

Non-abelian QFT over Pauli group

$$
\begin{aligned}
|0^m\rangle|0^n\rangle|0^n\rangle &\xrightarrow{\text{State prep}} \sum_{e\in\mathbb{F}_2^m} |e\rangle\,|0^n\rangle\,|0^n\rangle \\
&\xrightarrow{\text{Apply } X^{B^T e}} \sum_{e\in\mathbb{F}_2^m} |e\rangle X^{B^T e}\,|0^n\rangle\,|0^n\rangle \\
&\xrightarrow{\text{Decoder}} \sum_{e\in\mathbb{F}_2^m} X^{B^T e}\,|0^n\rangle\,|0^n\rangle \\
&\xrightarrow{\text{Apply "Fourier" } V^\dagger} \sum_{\lambda} \mathcal{P}(f(\lambda))|\lambda\rangle\,|\lambda\rangle\,.
\end{aligned}
$$

Further Hamiltonians

Further Hamiltonians

- Can the pilot state be prepared beyond log(n)-size connected components?
- Is there structure beyond sparsity in the commutation graph that can be exploited?
- Hardness of semi-classical spin glass?