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Lecture 6: Random Hamiltonians and the SYK Model

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1 Classical Random Hamiltonians

Random Hamiltonians have been studied for a long time in the classical setting, including in Wigner's surmise about spectra of heavy nuclei. We'll discuss the theory of spin glasses.

In the Curie-Weiss model of magnetism from the previous lecture, we had (with no external magnetic field) the Hamiltonian

$$H \propto \sum \sigma_i \sigma_j.$$

Instead, imagine a material with some defects, in which case the couplings between the spins might be some random matrix J_{ij}:

$$H = \frac{1}{\sqrt{N}} \sum_{i<j} J_{ij} \sigma_i \sigma_j.$$

By N we denote the number of particles, so the $\frac{1}{\sqrt{N}}$ normalization factor maintains an $O(N)$ scaling of the energy in the particle number. If additionally the J_{ij} are normally distributed, we refer to this as the **Sherrington-Kirkpatrick** (SK) model. The Gibbs distribution for a fixed choice of J looks like $\Pr(\vec{\sigma}) \propto \exp(H_J(\vec{\sigma}))$. In most cases (such as, perhaps, a fixed background lattice with impurities) it is not useful physically motivated of the J_{ij} as dynamical variables; we call this "quenched disorder."

We would like to compute the partition function

$$Z = \sum_{\vec{\sigma}} \exp\left(-\frac{\beta}{\sqrt{N}} \sum_{i<j} J_{ij} \sigma_i \sigma_j\right).$$

In particular, we generally want to look at $-\log Z$, because most physical quantities are easily expressed in terms of its derivatives. People tried to computed $-\log \mathbb{E}[Z]$, where the expectation value is over instances of J; we call this quantity the **annealed free energy**. It turns out this is only a good approximation in the large temperature limit, since otherwise the expectation value is dominated by extreme values of J. Instead, it is more helpful to compute $\mathbb{E}[-\log Z]$, the so-called "**quenched free energy**."

Edwards and Anderson used the "replica trick"

$$\log Z = \lim_{m \to 0} \frac{Z^m - 1}{m},$$

because powers are easier to work with and one would hope for the expectation value and limit to commute. This is particularly helpful because $\mathbb{E}[Z^2]$ is the partition function for two copies of the original system, provided that they always have the same disorder J_{ij}. So one can compute

$(Z^m - 1)/m$ for $m = 2, 3, \cdots$, and hope that the solution resembles some analytic function that they can evaluate as $m \to 0$.

This trick can be shown to lead to some strange predictions, such as negative entropies. Parisi gave a consistent appraoch and an exact formula, studying the "clustering" of subsystems of the Z^m system. He found that

$$\lim_{n \to \infty} \frac{1}{N} \mathbb{E}[E_0] \approx .763,$$

where E_0 is the ground state of the SK system.

In 2018, Montanari provided an algorithm which given J and some $\epsilon > 0$, provides a configuration $\vec{\sigma}$ such that the energy is within $1-\epsilon$ of the ground state, with time complexity $C(\epsilon)n^2$. Interestingly, computing the ground state exactly is NP-hard.

2 Random Quantum Hamiltonians

2.1 Language of the Following Models

In "second-quantization," we label multi-particle states by the number of particles in each state. Thus we can consider "creation" and "annihilation" operators a_i and $a_i^\dagger$ whose action on a state is to add a particle to that state. If these are for electrons, and the ground state is $|0\rangle$, then $a^\dagger |0\rangle$ is a state with a single electron. For fermions, we have the anticommutators $\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$, and $\{a_i, a_j^\dagger\} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

2.2 The SYK Model

For our purposes, we can rewrite the fermionic creation and annihilation operators in terms of Majorana fermions χ_i satisfying $\{\chi_i, \chi_j\} = \delta_{ij}$ and$\chi_i = \chi_i^\dagger$ (which physically means they are their own antiparticles!). In terms of these, our system of interest is the **SYK Model** (different names than the SK model),

$$H_{SYK} = \sum J_{ijkl} \chi_i \chi_j \chi_k \chi_l.$$

Here the coefficients J_{ijkl} are Gaussian with zero mean.

We're interested in the ground state energy. The operators $n_i := a_i^\dagger a_i$ are "number operators," in that their eigenstates are those with definite particle number and their eigenvalues are simply the particle numbers. We have $[n_i, n_j] = 0$. The idea is that by taking degree two combinations of the anticommuting χ fields, we get objects which commute. So writing the Hamiltonian in terms of the operators $\chi_i \chi_j$ and $\chi_k \chi_l$, we get something resembling the Hamiltonians in the previous section. These ideas were used to show that finding the ground state energy is NP-hard in the worst case and in fact QMA-hard as well.

By the way, the algebra of N Majorana fields can be realized on $\frac{N}{2}$ qubits by setting $\chi_1 = X \otimes I \otimes \cdots \otimes I$, $\chi_2 = Y \otimes I \otimes \cdots \otimes I$, $\chi_3 = Y \otimes X \otimes I \otimes \cdots \otimes I$, $\chi_4 = Y \otimes Z \otimes I \otimes \cdots \otimes I$, and so on.

Physics heuristics can approximate $E_0(H_{SYK}) \approx \frac{1}{2\sqrt{2}}\sqrt{n}$, and an algorithm can certify that $E_0 \leq \sqrt{1+\sqrt{6}}\sqrt{n}$. A quantum algorithm can output a state ρ such that $\mathrm{tr}(\rho H_{SYK}) = \Omega(\sqrt{n})$ (by the variational principle, this upper bounds the ground state energy). These states can be shown to necessarily highly entangled.

The idea behind this "Sum of Squares" algorithm is that if we want the bound $E_0 \leq \eta$, it suffices to show that $\eta I - H = \sum_i p_i^\dagger p_i$, where the p_i are polynomials in the Majorana fermion fields (this is because the p_i are Hermitian, so their squares are positive semidefinite).

If we write

$$H_{SYK} = \frac{1}{\sqrt{n}} \sum_i \chi_i \tau_i,$$

where $\tau_i := \sum_{j,k,l} \sqrt{n} J_{ijkl} \chi_j \chi_k \chi_l$, for a state ψ we can write

$$|\langle H_{SYK} \psi \rangle|^2 = \frac{1}{N} \left| \sum_i \langle \psi | \chi_i \tau_i | \psi \rangle \right|^2 \leq \cancel{\frac{1}{N} \left| \sum_i \langle \psi | \chi_i^2 | \psi \rangle \right|} \left| \sum_i \langle \psi | \tau_i^2 | \psi \rangle \right|.$$

The inequality is due to Cauchy-Schwartz, and the cancellation is because for Majorana fermions $\chi^\dagger = \chi$ and so the number operator is χ^2. Thus, the sum on i counts the number of particles. Because of some symmetry considerations (?), $\langle \psi | \tau_i^2 | \psi \rangle$ is degree 4. In paritcular, one can write

$$\sum_i \tau_i^2 = b_0 I + \sum_{i,j,k,l} b_{ijkl} \chi_i \chi_j \chi_k \chi_l$$

for some coefficients b_0, b_{ijkl}. By reasoning about the τ_i with some graph ideas, it seems a bound $E_0 \leq O(n)$ is achieved.

In a toy Hamiltonian, one might write

$$H = i(\chi_1 \chi_2 + \chi_3 \chi_4 + \cdots \chi_{n-1} \chi_n).$$

We had previously viewed Majorana fermions as the real and complex parts of other fermion fields, so if we pair them back up, this Hamiltonian is like the Hamming weight of a binary string. In this case, the ground state would be like $\rho = |11 \cdots 1\rangle\langle 11 \cdots 1|$ (I suppose what we're calling $|1\rangle$ and $|0\rangle$ here is a matter of notation). In terms of the Majorana fermion fields, this state is $\rho = (I - \chi_1 \chi_2)(I - \chi_3 \chi_4) \cdots$.

Back in the SYK model, our actual Hamiltonian is

$$H = i(\chi_1 \tau_1 + \chi_2 \tau_2 + \cdots).$$

Schematically, we would like our solution to look like $\rho = (I - \chi_1 \tau_1)(I - \chi_2 \tau_2) \cdots$, but this is not exactly right – it is not even clear that this is a valid density matrix. The solution is to adjoin n more Majorana fermions ($\frac{n}{2}$ qubits) $\sigma_1, \sigma_2, \cdots, \sigma_n$, write the previous expression with the σ_i in place of the τ_i, and then apply a unitary to "rotate" the τ_i back into place. That is,

$$\rho = e^{-it \sum_i t_i \sigma_i} (I - \chi_1 \sigma_1)(I - \chi_2 \sigma_2) \cdots (I - \chi_n \sigma_n) e^{it \sum_i t_i \sigma_i}.$$

"It's not at all clear why this should make any sense." It can be shown that some value of t gives a good approximation for the ground state energy.