Frontiers of Quantum complexity *Instructors: Anand Natarajan, Anurag Anshu*

Problem set 0

Due: Thu, Sept 11, 2025 *TFs: Yeongwoo Hwang, Tina Zhang*

Introduction: The questions below serve two purposes:

- To help examine your prior knowledge for the course. These are basic questions in the following topics: quantum algorithms, linear algebra, Pauli matrices.
- To survey your perspective on quantum theory and identify how it may differ from the approach presented in this course. This will help minimize any potential "language barriers."

This homework carries no score. Please provide your answer in the space below, and add your comments (if any).

Questions about your prior knowledge

If any of the questions are found to be unfamiliar, we recommend that you talk to us in the Office hours. While this course will encourage collaborations, we advise against collaborations for Homework 0 (as the goal is to test *your* prior knowledge).

1. **Finding a collision:** Suppose we are given an oracle O_f that evaluates a function $f : \{0,1\}^n \to \{0,1\}^{n-1}$. We are promised that the function is 2-to-1 (that is, every image has two pre-images) and we would like to find a collision - a pair of x, x' such that $f(x) = f(x')$.

   - Explain why this problem is harder than Simon's problem in a similar setting. Does that tell you at least how many queries are needed to solve the problem?
   - Show that Grover's search can be used to find a collision with $O(2^{n/2})$ queries.

   Answer:

2. **Quantum phase estimation and swap test:** Fix a basis $\{|1\rangle, |2\rangle, \ldots |d\rangle\}$ in $\mathcal{H}$. Then $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{d}$ is a basis for $\mathcal{H} \otimes \mathcal{H}$. Introduce the Swap unitary which acts on $\mathcal{H} \otimes \mathcal{H}$:

   $$\mathbf{Swap}(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle \ .$$

   Next, we introduce the famous swap test algorithm.

   Input: a quantum state on $\mathcal{H} \otimes \mathcal{H}$.

   - Add $|+\rangle_C$ and apply the unitary $|0\rangle\langle 0|_C \otimes \mathbb{1} + |1\rangle\langle 1|_C \otimes \mathbf{Swap}$.
   - Then apply $H_C \otimes \mathbb{1}$ and measure the register C in the computational basis $\{|0\rangle, |1\rangle\}$.

   Explain how you can view the swap test as an instance of the quantum phase estimation algorithm.

   Answer:

3. **Matrix Exponentiation:** Given Pauli operators $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, let $F(t) = e^{tX} Z e^{-tX}$. Write a closed form expression for the operator norm $\|F(t)\|_\infty$ as a function of t.

   Answer:

4. **Quantum entanglement:** Alice and Bob share the quantum state $|\psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |i\rangle_A \otimes |i\rangle_d$ on their joint Hilbert space $\mathcal{H} \otimes \mathcal{H}$, where $\mathcal{H}$ is d dimensional. Bob is 3 years old and likes to play with his system by applying unitaries. Suppose Bob applies a unitary U. Show that Alice can apply a unitary V such that $(V \otimes U)|\psi\rangle_{AB} = |\psi\rangle_{AB}$.

   Answer:

5. **Commutation between Pauli matrices:** Let X, Z be single qubit Pauli operators. For $a \in \{0,1\}^n$, we let $X^a = \otimes_{\ell=1}^n X_\ell^{a^{(\ell)}}$ and $Z^a = \otimes_{\ell=1}^n Z_\ell^{a^{(\ell)}}$. Given two n-qubit Pauli operators $O_1 = X^a Z^b$ and $O_2 = X^c Z^d$, we have $O_1 O_2 = p O_2 O_1$, where $p \in \{-1, 1\}$.

   - Write down an explicit expression for p in terms of a, b, c, d.
   - Under what conditions on a, b, c, d can you measure both O_1 and O_2 simultaneously?

   Answer:

Questions about your perspective on quantum theory

Answer the following questions based on your own understanding of quantum theory.

1. A quantum measurement does the following.

   (a) It collapses a pure quantum state to a different pure quantum state in a probabilistic manner.

   (b) It destroys the quantum state and just the classical information remains.

   (c) It maps a pure quantum state to a mixed quantum state.

   (d) It simply updates our quantum description of a system.

   (e) I am not familiar with quantum measurements

   Answer or comments:

2. **Quantum state:** The interpretation of quantum state can vary greatly across physicists and computer scientists. Which of the following accurately describes a quantum state?

   (a) A quantum state is a unit vector in a vector space.

   (b) A quantum state is a positive-semidefinite matrix with trace one.

   (c) Probability distributions cannot be viewed as quantum states.

   (d) A quantum state is merely a mathematical description, and does not physically exist.

   (e) I am not familiar with quantum states.

   Answer or comments: